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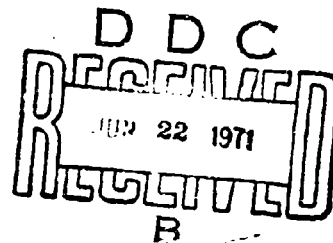
REPORT I

A THEORY OF VISCOPLASTICITY
WITHOUT A YIELD SURFACE
PART I - GENERAL THEORY

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13. ABSTRACT

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The underlying principle is that the history of deformation is defined in terms of a "time scale" which is not measured by a clock, but is in itself a property of the material at hand.

The theory is unifying in the sense that theories of plasticity, viscoelasticity and elasticity can be obtained from it as special cases by imposing suitable constraints on the material parameters involved; furthermore, it does not make use of the idea of a yield surface.

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Section 1. Endochronic Theory of Viscoplasticity

In current theories of plasticity, to explain the observed discontinuities in material behavior upon loading beyond the "yield point" and upon unloading, one has to introduce the concept of a yield surface in stress space as well as a "loading function" to distinguish between loading and unloading. Similarly, in the case of viscoplasticity, the existence of a static stress-strain relation and a yield surface are assumed and the stress increment, with respect to the static value, is related to the strain rate, or more generally to the strain history, by a constitutive equation.

However, the fact that the phenomenon of yield is usually a gradual transition from a linear to a non-linear stress-strain response, makes it difficult to say precisely where yield has occurred, to the extent that different definitions of yield are used for this purpose. Three such definitions, for instance, are (a) the deviation from linearity in the relation between some measure of strain and stress (b) the intersection of the initial part of a stress strain curve and the backward linear extrapolation of the "plastic" part of the curve and (c) a value of "proof" stress corresponding to an arbitrarily defined value of "proof" strain.

Though, from an engineering viewpoint, the initial yield surface is not overly influenced by the definition of yield, it has been found experimentally that subsequent yield surfaces of a strain hardening material are influenced by the definition of yield to an extraordinary degree. (See Appendix I). If we insist that the increment in plastic strain is to be normal to the yield surface, then, for complex stress histories, each such definition will give rise to a different plastic strain history. Only one of these can be the correct one.

The conceptual difficulties that are encountered by the introduction of the yield surface are completely circumvented by our theory of plasticity which is developed on the basis of the observation that the state of stress in the neighborhood of a point in a plastic material depends on the set of all previous states of deformation of that neighborhood, but it does not depend on the rapidity at which such deformation states have succeeded one another*.

The independence of stress of the rapidity of succession of deformation states is achieved by introducing a time scale ξ which is independent of t , the external time measured by a clock, but which is intrinsically related to the deformation of the material.

Of course there are many ways of introducing such a time scale. However, it appears almost mandatory that ξ should be a monotonically increasing function of deformation, otherwise two different states of deformation could exist "simultaneously" i.e. for the same value of ξ . Furthermore, a positive rate of change $\frac{d\epsilon}{d\xi}$, of the internal energy density ϵ with respect to ξ could not be interpreted unambiguously as a process of increasing ϵ , if $d\xi$ could be negative.

A logical definition** for ξ is then given by the relation

*In the present Section and in subsequent Sections (with the exception of Section 2) we shall assume that mechanical changes take place in a constant temperature environment, such as an isothermal atmosphere. The thermal changes in the material will, therefore, be mechanically induced and, in general, will remain small. Conversely, only thermal changes of this nature will be considered in this paper.

**Alternative but less general definitions have appeared in the literature. For instance, Ilyushin⁽¹⁾ and later Rivlin⁽²⁾ defined a "time" s by the relation $ds^2 = dC_{ij}dC_{ij}$. However, we have found that this definition is too narrow to describe, quantitatively, material behavior in the plastic range as will be discussed later. The effect of temperature on ξ , will at this time, be considered sufficiently small to be negligible. For a more vague allusion to this possibility see also, Schapery (3).

$$d\xi^2 = P_{ijkl} dC_{ij} dC_{kl} \quad (3.1)$$

where C_{ij} is the Right Cauchy-Green tensor and P_{ijkl} is a fourth order tensor which could depend on C_{ij} . The positive definite nature of $d\xi^2$ requires that P_{ijkl} be positive definite. In the case of small deformation

$$d\xi^2 = P_{ijkl} d\epsilon_{ij} d\epsilon_{kl}$$

where ϵ_{ij} is the small deformation strain tensor and P_{ijkl} could depend on ϵ_{ij} .

Actual materials, on the other hand, do, in general, depend on the history of deformation as well as on the rapidity, or rate, at which deformation states succeed one another. To describe materials of this type one may construct a theory of viscoplasticity by introducing a time scale ζ which is related to the external time t .

It appears logical to define ζ by the relationship

$$d\zeta^2 = \alpha^2 d\xi^2 + \beta^2 dt^2$$

where α and β are scalar material parameters. Henceforth $d\zeta$ will be called an "intrinsic time measure", and $z(\zeta)$, such that $\frac{dz}{d\zeta} > 0$ ($0 \leq \zeta < \infty$), will be called an "intrinsic time scale".

In our theory, the stress (among other properties) is necessarily, a functional of the strain history, defined with respect to the intrinsic time scale, the latter being a property of the material at hand. As a result we have called our theory an endochronic theory of viscoplasticity.

The theory will now be developed in a general thermodynamic framework in Section 3. Before this is done, however, the thermodynamic foundations are laid in Section 2.

Section 2. Thermodynamic Foundations

The following are the fundamental laws of thermodynamics, which apply to all continuous media irrespective of their constitutive properties. (For materials that are solid-like, in the sense that they have a memory of their initial configuration, it is more convenient to express these laws in the material coordinate system x^i). In differential form, these are the first law of thermodynamics,

$$\dot{e} = (\rho_0/2\rho) \tau^{ij} \dot{C}_{ij} - h^i_{,i} + \dot{Q}, \quad (2.1)$$

the rate of dissipation inequality,

$$\theta \dot{\gamma} = (\rho_0/2\rho) \tau^{ij} \dot{C}_{ij} - \dot{\psi} - \eta \dot{\theta} \geq 0 \quad (2.2)$$

and the heat conduction inequality

$$-h^i \theta_{,i} \geq 0. \quad (2.3)$$

The symbols in eq.'s (2.1), (2.2) and (2.3) have the following meaning: $\rho_0 e$ is the internal energy per unit mass; ρ_0 and ρ are the initial and current mass densities respectively; τ^{ij} is the stress tensor in the material coordinate system x^i ; C_{ij} is the right Cauchy-Green tensor, h^i is the heat flux vector per unit undeformed area in the material system; $\rho_0 Q$ is the heat supply per unit mass; θ is the temperature, γ the irreversible entropy and ψ and η are the free energy and entropy, respectively, per unit undeformed volume finally a subscript following a comma denotes differentiation with respect to the corresponding material coordinate. A dot over a quantity denotes material derivative with respect to time. To avoid repetitious statements, henceforth we shall refer to C_{ij} as the "deformation".

In the case of dissipative materials the stress, the internal energy and entropy densities (and, therefore, the free energy density) of a material neighborhood depend on the entire history of deformation and temperature of that neighborhood.

In the theory of irreversible thermodynamics the effects of history are taken into account by specifying that the stress and free energy density are functions of the current values of C_{ij} and θ as well as n additional independent variables q_a , not necessarily observable, called "internal variables". These may be scalars or components of vectors or tensors in the material frame; whatever their geometric nature they must remain invariant with translation and rotation of the spatial system to satisfy the principle of material indifference. Thus:

$$\psi = \psi(C_{ij}, \theta, q_a) \quad (2.4)$$

$$\tau^{ij} = \tau^{ij}(C_{kl}, \theta, q_a) \quad (2.5)$$

It has been shown elsewhere⁽⁴⁾ that

$$\tau^{ij} = \frac{2\rho}{\rho_0} \frac{\partial \psi}{\partial C_{ij}} \quad (2.6)$$

$$\eta = - \frac{\partial \psi}{\partial \theta} \quad (2.7)$$

$$\theta \dot{\gamma} = - \frac{\partial \psi}{\partial q_a} \dot{q}_a \geq 0 \quad (2.8)$$

Furthermore the heat flux vector h^i is a function of $\theta_{,i}$, θ , C_{ij} and q_a i.e.,

$$h^i = h^i(\theta_{,i}, \theta, C_{kl}, q_a) \quad (2.9)$$

subject to the conditions:

$$h^i \Big|_{\theta, i = 0} = 0, \quad h^i \theta_{, i} \leq 0. \quad (2.10 \text{ a, b})$$

Finally eq. (2.1) in conjunction with eq.'s (2.6) and (2.7) yields

$$h^i_{, i} = \theta \left(\frac{\partial \psi}{\partial \theta} \right) - \frac{\partial \psi}{\partial q_\alpha} \dot{q}_\alpha + \dot{Q} \quad (2.11)$$

The remarkable property of the above equations is that they apply to all materials irrespective of their constitution. This has not been generally recognized. In fact the constitutive nature of the material follows from the constitutive properties of q_α . For example, in elastic materials $q_\alpha = 0$, whereas in viscoelastic materials q_α are given by a set of differential equations of the type,

$$\dot{q}_\alpha = f_\alpha(q_\beta, c_{ij}, \theta) \quad (2.12).$$

The question of how q_α are determined for viscoplastic materials is considered in the next Section.

Section 3. Constitutive Equations in Viscoplasticity

From the right hand side of eq. (2.8) and the fact that $\frac{dz}{dt} > 0$ and $dz/dt > 0$, it follows that

$$-\frac{\partial \psi}{\partial q_a} \frac{dq_a}{dz} \geq 0 \quad (a \text{ not summed}) \quad (3.1)$$

where the inequality is valid unless $\frac{dq_a}{dz} = 0$. It also follows from inequality (3.1) that $\frac{dq_a}{dz}$, q_a , C_{ij} and θ must be related otherwise $\frac{\partial \psi}{\partial q_a}$ and $\frac{dq_a}{dz}$ could be prescribed independently and in such a fashion, that the inequality would be violated. In this event there must exist a set of relations

$$\frac{dq_a}{dz} = f_a(C_{ij}, q_b, \theta) \quad (3.2)$$

for all a , where the functions f_a are material functions.

It must be noted that, as a result of eq. (3.2) q_a are indeed functionals of the histories of deformation and temperature with respect, however, to the intrinsic time scale z which is, itself, a material property.

Thus, at least formally, the constitutive equations of the endochronic theory of viscoplasticity are now complete in the sense that given the material functions f_a, ψ and h^i then for some specified deformation and temperature histories, q_a are found from eq. (3.2) and thus τ^{ij} and n are found from eq's. (2.6) and (2.7) respectively; similarly h^i the heat flux vector is determined from eq. (2.9).

Ideally, one would like to know the thermomechanical three-dimensional response of a material over the whole spectrum of mechanical and thermal conditions, i.e., under all variations in strain, strain rate (or more generally, history of strain) and temperature. However, such a task would be a momentous, if not an impossible, undertaking; the experimental evaluation of the material

functions involved, under wide conditions of strain and temperature, would be impractical.

Fortunately the domain of specification of design conditions is usually limited in some way; for instance usually, (a) large changes of temperature, fast rates of loading, but small strains are prescribed; or (b) small changes in temperature and small rates of loading but large strains and/or displacements prevail. More extreme mechanical as well as thermal conditions are rarer.

It is reasonable to expect that material behavior would be easier to describe mathematically over a narrower domain of environmental conditions, where the applicability or "correctness" of such mathematical formulation would be easier to check experimentally.

In what follows we shall consider situations in which the strain in a material region R as well as the temperature changes relative to a uniform reference temperature θ_0 are "small". To make the above statement more precise let $\epsilon_{ij}(z')$ denote the history of the strain tensor^{*}, for $z_0 \leq z' \leq z$ where z_0 is some initial intrinsic time. Set

$$||\epsilon_{ij}(z')|| \equiv \{\epsilon_{ij}(z') \epsilon_{ij}(z')\}^{1/2} \quad (3.3)$$

and let the supremum of $||\epsilon_{ij}(z')||$ be Δ .

Similarly let $\vartheta(z')$ be the history of the temperature increment relative to the reference temperature θ_0 and let the supremum of $|\vartheta(z')|$ be δ . The notion of smallness is made precise by stipulating that $\Delta \ll 1$ and $\delta \ll 1$.

Thus, formally

$$\theta = \theta_0 + \vartheta \quad (3.4)$$

$$* \epsilon_{ij} = \frac{1}{2} (C_{ij} - \delta_{ij})$$

$$\eta = \eta_0 + \chi \quad (3.5)$$

$$\|\epsilon_{ij}(z')\|_{\text{sup}} = \Delta, \quad |\mathcal{D}(z')|_{\text{sup}} = \delta \quad (3.6)$$

To complete the formalism let χ be the entropy change relative and a reference uniform entropy η_0 , and let σ_{ij} denote the stress tensor. The reference state is defined by the condition that $\sigma_{ij} = 0$, $\psi = \mathcal{D} = \chi = 0$, $q_\alpha = 0$.

Under these conditions, eq.'s (2.4) through (2.9) and eq. (2.10b) become,

$$\psi = \psi(\epsilon_{ij}, \mathcal{D}, q_\alpha) \quad (3.7)$$

$$\sigma_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}} \quad (3.8)$$

$$\chi = - \frac{\partial \psi}{\partial \mathcal{D}} \quad (3.9)$$

$$\theta_0 \dot{\gamma} = - \frac{\partial \psi}{\partial q_\alpha} \dot{q}_\alpha \geq 0 \quad (3.10)$$

$$h_i = k_{ij} \mathcal{D}_{,j} \quad (3.11)$$

$$h_i \mathcal{D}_{,i} \leq 0 \quad (3.12)$$

where k_{ij} is the thermal conductivity tensor.

Finally eq. (2.11), becomes

$$h_{i,i} = \theta_0 \left(\frac{\partial \psi}{\partial \mathcal{D}} \right)' - \frac{\partial \psi}{\partial q_\alpha} \dot{q}_\alpha + \dot{Q} \quad (3.13)$$

It is shown in Appendix II that $|q_\alpha|$ and $|\dot{q}_\alpha|$ may stay small in the sense that given two positive numbers δ_1^α and δ_2^α , however small, Δ and δ can be chosen small enough such that $|q_\alpha| < \delta_1^\alpha$ and $|\dot{q}_\alpha| < \delta_2^\alpha$.

At this stage one may obtain the corresponding equation for q_α by linearizing eq. (3.2). However it is physically more meaningful and, a posteriori, more rewarding to examine more closely the rate of change of

irreversible entropy $\dot{\gamma}$.

From eq. (3.10),

$$\theta_0 \frac{d\gamma}{dz} = - \frac{\partial \psi}{\partial q_a} \frac{dq_a}{dz} \geq 0 \quad (3.14)$$

It follows from eq.'s (3.14) and (3.2) that $\frac{d\gamma}{dz}$ may be expressed as a function of $\frac{dq_a}{dz}$, ϵ_{ij} and \mathcal{D} , subject to the condition that $\frac{d\gamma}{dz} = 0$ whenever $\frac{dq_a}{dz} = 0$ for all a . Thus, if we expand $\theta_0 \frac{d\gamma}{dz}$ in a Taylor series and ignore terms of order higher than $O(\delta^2)^*$ and observe the inequality (3.10), thereby eliminating the linear terms in the expansion, then

$$\theta_0 \frac{d\gamma}{dz} = b_{\alpha\beta} \frac{dq_\alpha}{dz} \frac{dq_\beta}{dz} \quad (3.15)$$

Eq.'s (3.14) and (3.15) are simultaneously satisfied if

$$\frac{\partial \psi}{\partial q_a} + b_{\alpha\beta} \frac{dq_\alpha}{dz} = 0 \quad (3.16)$$

With eq. (3.16) the constitutive description of a viscoplastic material is now complete.

Explicit Constitutive Equations

Explicit constitutive equations for viscoplastic materials under conditions of small strain and small changes in temperature are obtained by expanding ψ in eq. (3.7) in Taylor Series and omitting terms of order higher than $O(\delta^2)$; * linear terms must vanish to satisfy the initial conditions.

Before the expansion is carried out, however, it appears desirable to regard q_a not as scalars but components of second order tensors. This, as will be shown, obviates certain difficulties which arise with the representation of fourth order tensors. For instance, in Ref. (4), we were faced

* δ_a is the largest of δ_1^a and δ_2^a . Also, $\hat{\delta}$ is the largest of δ , Δ and δ_1^a .

with having to assume, without proof, that a fourth order tensor C_{ijkl} such that,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$$

is given by the series

$$C_{ijkl} = \sum_a \frac{a_{ij} a_{kl} a_a}{a_a} \quad (3.17)$$

where $a_{ij\alpha}$ are second order symmetric tensors and a_α are scalars. Problems such as this are obviated by giving the internal variables a tensorial character. Thus the free energy density and other thermodynamic quantities are now functions of ϵ_{ij} , θ and n internal variables q_{ij}^α ($\alpha = 1, 2, \dots, n$), where q_{ij}^α are symmetric second order tensors, with respect to the material system x_i . In this notation, eq.'s (3.13) through (3.16) now read,

$$h_{i,i} = \theta_0 \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial q_{ij}^\alpha} \dot{q}_{ij}^\alpha + \dot{Q} \quad (3.18)$$

$$\theta_0 \frac{d\gamma}{dz} = - \frac{\partial \psi}{\partial q_{ij}^\alpha} \frac{dq_{ij}^\alpha}{dz} \geq 0 \quad (3.19)$$

$$\theta_0 \frac{d\gamma}{dz} = \sum_a b_{ijkl}^\alpha \frac{dq_{ij}^\alpha}{dz} \frac{dq_{kl}^\alpha}{dz} \quad (3.20)^*$$

and

$$\frac{\partial \psi}{\partial q_{ij}^\alpha} + b_{ijkl}^\alpha \frac{dq_{kl}^\alpha}{dz} = 0 \quad (3.21)$$

(α not summed)

*Expansions of the type $b_{ijkl}^{\alpha\beta} \frac{dq_{ij}^\alpha}{dz} \frac{dq_{kl}^\beta}{dz}$ and $A_{ijkl}^{\alpha\beta} q_{ij}^\alpha q_{kl}^\beta$ reduce to the above form. See Ref. 5.

Furthermore, in view of my previous discussion,

$$\begin{aligned} \psi = & \frac{1}{2} A_{ijkl} \epsilon_{ij} \epsilon_{kl} + B_{ijkl}^{\alpha} \epsilon_{ij} q_{kl}^{\alpha} + \frac{1}{2} C_{ijkl}^{\alpha} q_{ij}^{\alpha} q_{kl}^{\alpha} \\ & + D_{ij} \vartheta \epsilon_{ij} + E_{ij}^{\alpha} \vartheta q_{ij}^{\alpha} + \frac{1}{2} F \vartheta^2 \end{aligned} \quad (3.22)$$

Though, in principle, eq.'s (3.8), (3.9), (3.18), (3.21) and (3.22) are sufficient for the derivation of explicit constitutive equations, we shall obtain these only for isotropic material, so as to keep the algebra at a minimum. For such materials

$$\begin{aligned} A_{ijkl} &= A_1 \delta_{ij} \delta_{kl} + A_2 \delta_{ik} \delta_{jl} \\ B_{ijkl}^{\alpha} &= B_1^{\alpha} \delta_{ij} \delta_{kl} + B_2^{\alpha} \delta_{ik} \delta_{jl} \\ C_{ijkl}^{\alpha} &= C_1^{\alpha} \delta_{ij} \delta_{kl} + C_2^{\alpha} \delta_{ik} \delta_{jl} \\ D_{ij} &= D \delta_{ij} \\ E_{ij}^{\alpha} &= E^{\alpha} \delta_{ij} \\ k_{ij} &= k \delta_{ij} \\ b_{ijkl}^{\alpha} &= b_1^{\alpha} \delta_{ij} \delta_{kl} + b_2^{\alpha} \delta_{ik} \delta_{jl} \end{aligned} \quad (3.23 \text{ a-f})$$

It is worth noting that here we consider materials which are "stable" in the sense that straining of the reference configuration under isothermal conditions will increase the free energy density ψ . Thus A_{ijkl} and C_{ijkl}^{α} are positive definite. As a consequence A_1 , A_2 , C_1^{α} and C_2^{α} are all positive.

Omitting superfluous algebra, the coupled thermomechanical constitutive equations take the following form in terms of the hydrostatic stress $\sigma = \frac{\sigma_{kk}}{3}$, the deviatoric stress tensor σ_{ij} , the increment in temperature ϑ , the

hydrostatic strain ϵ_{kk} , the deviatoric strain tensor e_{ij} and the entropy increment χ ; in terms of the above notation:

$$s_{ij} = 2 \int_{z_0}^z \mu(z-z') \frac{\partial e_{ij}}{\partial z'} dz' \quad (3.24)$$

$$\int_{z_0}^z K(z-z') \frac{\partial \epsilon_{kk}}{\partial z'} dz' + \int_{z_0}^z D(z-z') \frac{\partial \vartheta}{\partial z'} dz' \quad (3.25)$$

$$-\chi = \frac{\partial \psi}{\partial \vartheta} = \int_{z_0}^z D(z-z') \frac{\partial \epsilon_{kk}}{\partial z'} dz' + \int_{z_0}^z F(z-z') \frac{\partial \vartheta}{\partial z'} dz' \quad (3.26)$$

where,

$$2 \mu(z) = \left(A_2 - \sum_{\alpha} \frac{B_2^{\alpha} B_2^{\alpha}}{C_2^{\alpha}} \right) H(z) + \sum_{\alpha} \frac{B_2^{\alpha} B_2^{\alpha}}{C_2^{\alpha}} e^{-\rho_{\alpha} z} \quad (3.27)$$

$$K(z) = \left(A_0 - \sum_{\alpha} \frac{B_0^{\alpha} B_0^{\alpha}}{C_0^{\alpha}} \right) H(z) + \sum_{\alpha} \frac{B_0^{\alpha} B_0^{\alpha}}{C_0^{\alpha}} e^{-\lambda_{\alpha} z} \quad (3.28)$$

$$D(z) = \left(D - \sum_{\alpha} \frac{B_{\alpha} E_{\alpha}}{C_{\alpha}^{\alpha}} \right) H(z) + \sum_{\alpha} \frac{B_{\alpha} E_{\alpha}}{C_{\alpha}^{\alpha}} e^{-\lambda_{\alpha} z} \quad (3.29)$$

$$F(z) = \left(F - \sum_{\alpha} \frac{E_{\alpha} E_{\alpha}}{C_{\alpha}^{\alpha}} \right) H(z) + \sum_{\alpha} \frac{E_{\alpha} E_{\alpha}}{C_{\alpha}^{\alpha}} e^{-\lambda_{\alpha} z} \quad (3.30)$$

$$A_0 = 1/3 (3A_1 + A_2), \quad B_0 = 1/3 (3B_1 + B_2), \quad C_0 = 1/3 (3C_1 + C_2) \quad (3.31)$$

$$\rho_{\alpha} = \frac{C_2^{\alpha}}{b_2^{\alpha}}, \quad \lambda_{\alpha} = \frac{C_0^{\alpha}}{b_0^{\alpha}} \quad (3.32)$$

$$b_0^{\alpha} = 3b_1^{\alpha} + b_2^{\alpha} \quad (3.33)$$

The heat conduction equation is similarly found to be:

$$\frac{dt}{dz} k \vartheta_{,ii} = \frac{\partial}{\partial z} \int_{z_0}^z C_v(z-z') \frac{\partial \vartheta}{\partial z'} dz' - \theta_0 \frac{\partial}{\partial z} \int_{z_0}^z D(z-z') \frac{\partial \epsilon_{kk}}{\partial z'} dz'$$

$$+ \hat{Q} + \sum_a b_1^a \hat{q}_{ii}^a \hat{q}_{jj}^a + \sum_a b_2^a \hat{q}_{ij}^a \hat{q}_{ij}^a \quad (3.34)$$

where

$$C_V(z) = -\theta_0 F(z) \quad (3.35)$$

and a $\dot{}$ over a quantity implies differentiation with respect to z . The lower limit z_0 denotes the intrinsic time of the reference state.

Section 4. Endochronic Theory of Plasticity and its Relation to Present Theories

Our theory of plasticity, which is a rate independent endochronic theory, is obtained by replacing the time measure $d\zeta$ by $d\xi$. The time scale now becomes $z(\xi)$, but the form of the constitutive equations remains unaltered. In particular, the "linear" form of our theory is obtained by setting

$$d\xi^2 = p_{ijkl} d\epsilon_{ij} d\epsilon_{kl} \quad (4.1)$$

where p_{ijkl} is a positive definite fourth order material tensor. We repeat the constitutive equations of the linear theory, in the particular case when the deformation is isothermal so that a comparison may be made with current theories. When $\Theta \equiv 0$, then eq.'s (3.24) and (3.25) become,

$$s_{ij} = 2 \int_{z_0}^z \mu(z-z') \frac{\partial e_{ij}}{\partial z'} dz' \quad (4.2)$$

$$\sigma_{kk} = 3 \int_{z_0}^z K(z-z') \frac{\partial \epsilon_{kk}}{\partial z'} dz' \quad (4.3)$$

where $z = z(\xi)$.

If the material behaves elastically under pressure (so-called plastically incompressible) then $K(z)$ is a constant and in this case

$$\sigma_{kk} = 3K\epsilon_{kk} \quad (4.4)$$

Whereas,

$$s_{ij} = 2 \int_{z_0}^z \mu(z-z') d\epsilon_{ij}(z') \quad (4.5)$$

Let now $\mu(z)$ consist of a single exponential term i.e.

$$\mu(z) = \mu_0 e^{-az} \quad (4.6)$$

In this event

$$s_{ij} = 2\nu_0 \int_{z_0}^z \frac{1}{z} \frac{d}{dz} (z - z') de_{ij}(z') \quad (4.7)$$

The integral eq. (4.7) is reducible to the differential equation

$$de_{ij} = \frac{a}{2\nu_0} dz s_{ij} + \frac{1}{2\nu_0} ds_{ij} \quad (4.8)$$

Now, $\frac{1}{2\nu_0} ds_{ij}$ may be identified as the "elastic" component of deviatoric strain of classical plasticity. If one follows the traditional definition of "plastic strain" de_{ij}^P given below, i.e.,

$$de_{ij}^P = de_{ij} - de_{ij}^e \quad (4.9)$$

then in view of eq. (4.8)

$$de_{ij}^P = \frac{a}{2\nu_0} dz s_{ij} \quad (4.10)$$

But these are the Prandtl-Reuss relations. Hence our present theory contains these relations as a special case. Where then does it differ from this theory? It does in the interpretation of the proportionality coefficient dz . In the Prandtl-Reuss theory $d\xi$ may be positive negative or zero, and in fact, z has been identified with the yield surface, i.e., plastic action is assumed to occur when $dz > 0$, where

$$z = z(s_{ij}) \quad (4.11)$$

but that de_{ij}^P is zero whenever

$$dz \leq 0 \quad (4.12)$$

In the present theory dz is always positive if the material is deforming (it is zero only when deformation does not take place). Thus always,

$$dz \geq 0 \quad (4.13)$$

Furthermore dz is not given by eq. (4.11) i.e. it is not related to some yield surface but its definition is entirely kinematic. Thus, no yield phenomenon or surface are postulated here. One obtains the stress response by merely monitoring the history of strain.

Also the theory admits a further generality since $u(z)$ need not consist of a single exponential term.

For instance $u(z)$ may be of the form

$$u(z) = u_0 + u_1 e^{-\alpha z} \quad (4.14)$$

In this case, however, the differential form of eq. (4.7) becomes:

$$2(u_0 + u_1) de_{ij} + 2u_0 \alpha e_{ij} dz = ds_{ij} + \alpha s_{ij} dz \quad (4.15)$$

The shear modulus μ , at $z = 0$, (initial modulus), is $(u_0 + u_1)$. The "plastic" components of the deviatoric shear strain tensor are given from eq. (4.9), i.e.,

$$de_{ij}^p = \frac{\alpha}{2 u_1} dz s_{ij} - 2 u_0 e_{ij} \quad (4.16)$$

Note that eq. (4.16) does not satisfy the Prandtl-Reuss relations, which are also violated if one adds more exponential terms to the right hand side of eq. (4.14). In fact these relations will be satisfied if and only if $u(z)$ is given by eq. (4.6), i.e., u is represented by a single exponential term only. This situation is not particularly disturbing. Peters et Als⁽⁴⁰⁾ carried out experiments on thin walled 14S-T4 aluminum alloy cylinders by loading these in combined compression and torsion and found that the Prandtl-Reuss relations were not satisfied, for this particular metal.

Conclusions

A theory has been presented here, the scope of which is wide enough to allow a rational phenomenological description of mechanical behavior of materials under various histories of strain and temperature. In particular, the viscoplastic behavior of materials is formulated mathematically, without recourse to the dichotomy of the deformation history in plastic and elastic parts and without the necessity of introducing discontinuities in material behavior, such as yield surfaces.

The theory merely asserts that, to every history of deformation gradient and temperature of a neighborhood there corresponds a unique state of stress in that neighborhood. An entirely novel feature of the theory is that these histories are defined with respect to a time scale, which itself is a material property.

In this paper, we have merely presented the framework of the theory without actually evaluating the material functions involved, through the use of experimental data. This, however, will be done in Part II of this paper, where it will be shown that the theory describes experimentally observed plastic behavior of metals with a remarkable degree of accuracy.

Stipulation

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Appendix 1

The following is a short account of the experimental work on (a) the effect of the definition of yield on the shape of the yield surface and (b) of the work on viscoplasticity. The references given are by no means exhaustive and the author wishes to apologize to people of whose work he is not currently aware.

In Ref.'s 6 and 7, aluminum alloy tubes** were subjected to shear pre-strain by twisting well into the plastic region by a predetermined amount. The yield surface corresponding to this degree of prestrain was established by loading the tubes in combined tension and torsion.

In Ref. 6, Naghdi found that subsequent yield surfaces distorted in the direction of the shear axis with a pronounced Bauschinger effect in shear but there was no effect on the yield stress in tension (i.e. the yield locus did not change in the vicinity of zero shear stress).

On the other hand, in Ref. 7, Ivey observed that the yield surface, in addition to distortion, underwent a large amount of translation in the direction of the shear axis, so that for large prestrains the origin of the stress space was outside the yield surface. However, he was in agreement with Naghdi in that the presence of shear prestrain did not affect the yield stress in tension. Both authors used deviation from linearity in a stress strain diagram as a definition of yield.

Mair and Pugh⁽⁸⁾ to check the absence of "cross effect", carried out their own experiments on copper with a high degree of isotropy. However they used a different definition of yield, this being the point of intersection of the initial straight part of the stress-strain curve with a backward linear extrapolation of the "plastic" part of the stress-strain curve.

Their results varied significantly from those of Ivey and Naghdi. They found that expansion and distortion of the initial locus took place with a strong cross effect between shear and tension. Also a pronounced Bauschinger effect in torsion was found with large initial positive pretorsion. These authors also observed pronounced "plastic" unloading in shear.

The results of Szczepinski and Miastkowski⁽⁹⁾ tend to confirm the findings of Mair and Pugh⁽⁸⁾. Their results, moreover, were significant in other respects. Specifically, using the proof strain to define yield, they studied aluminum alloy sheets under biaxial tension with the intention of finding the effect of prestrain on the shape of the yield surfaces. They observed, migration, distortion, expansion and sometimes rotation of the initial yield locus.

Similar conclusions* can be drawn from Szczepinski's paper⁽¹⁰⁾, as well as Miastkowski and Szczepinski's⁽¹⁷⁾, in which tubular brass specimens were subjected to combined axial and circumferential stress.

Initial and subsequent yield loci were plotted when yield was defined (a) as departure from linearity or (b) when it was set to correspond to a certain proof strain. In particular, when definition (a) was used, subsequent yield loci did not contain the initial locus, but when (b) was used, with proof strain set at 0.5% subsequent loci contained the initial locus.

Attempts to describe the change of the yield locus with prestrain, by simple models have not proved satisfactory. Batdorf and Budianski⁽¹³⁾, suggested that after prestrain, the yield locus is the minimum surface

* In this connection, see also work of the same general nature by Bertsch and Findley⁽¹¹⁾ and Hu and Bratt⁽¹²⁾.

through the point of prestrain and the initial yield locus. This model however does not account for the Bauschinger effect. The kinematic hardening rule** proposed by Prager⁽¹⁴⁾, was partially successful, in so far as it can be of value only when the stress-strain curve of a material in simple tension is bilinear⁽¹⁵⁾. Otherwise subsequent shapes of yield locus must be defined in terms of a parameter that depends on the history of strain⁽¹⁵⁾ to obtain realistic unloading behavior.

A more realistic model is the one by Hodge⁽¹⁶⁾ which includes translation, expansion and distortion of the yield surface. This model covers all contingencies but does not include the history of stress on the shape and position of the yield locus.

However, every definition of yield gives rise to a different yield surface. If we insist that the increment of plastic strain is to be normal to the yield surface, then, for a complex but specific loading history, each such definition will give rise to a different plastic strain history. Only one of these can be the correct one.

So it appears that through Eisenberg's and Phillip's⁽¹⁵⁾ mathematical description of a yield surface has been most promising, we must be prepared to question, if necessary, whether the concept of yield point and yield surface are the only way by which plastic effects may be described, especially in view of the fact that these may take place immediately following the initiation of deformation of material, though they may be negligible in the region of small strains. This would agree with the point of view that dislocations (and, therefore, plastic behavior) originate immediately upon initiation of the loading.

** See also Ref. (19).

Viscoplasticity

The need for the development of the theory of viscoplasticity arises from the recognition of the strain rate sensitivity of metals under dynamic loading.

The difficulty in trying to synthesize a rational "rate" theory from experimental observations, a priori, lies in the fact that under dynamic conditions the inertia effects are significant. In the absence of a constitutive theory, these effects cannot be calculated.* Therefore, in the case of dynamic theories, such as viscoplasticity, theory and experiment must advance together.

The literature abounds with data on the subject of strain rate sensitivity, particularly in one dimension.⁽²⁰⁻³⁰⁾ Lindholm⁽³²⁾ carried out dynamic experiments in one and two dimensions in an attempt to generalize results which were arrived at, by consideration of thermally activated processes and their relation to dislocation theory in metals. See Ref.'s 35-40.

An early attempt at a theoretical viscoplastic constitutive equation in one dimension is due to Malvern.^(31,32) This equation assumes the existence of a "static" stress-strain relation and then relates the stress increment, with respect to the static value, to the strain rate.

Modifications and generalizations of Malvern's equation were made by Lubliner⁽³³⁾ who included a limiting maximum stress-strain curve, and by Perzyna⁽³⁴⁾, Perzyna and Wojno⁽³⁵⁾ who proposed a multiaxial generalization

* Constant strain rate experiments would appear to be an exception, by being less susceptible to inertia effects. However, Ref. 38 tends to negate this. Long specimens give different responses to short ones, under the same conditions.

for finite strains, assuming the additivity of the elastic and plastic strain components and by Perzyna⁽³⁹⁾ who used concepts of internal coordinates and irreversible thermodynamics to eliminate the above assumptions and to put the theory on sounder foundations.

Though, in his last treatment, Perzyna⁽³⁹⁾ abandoned the additivity of plastic and elastic strains, he still retained the concepts of yield stress (and yield surface) and the hypothesis of a datum plastic stress strain relation, with respect to which "strain history" is to be related to the "excess" stress through the internal coordinates.

Our theory differs from Perzyna's theory in this respect.

We close by mentioning that, with the exception of the papers by Perzyna⁽³⁹⁾, only a moderate research effort has been made in the area of coupling between a viscoplastic and a thermal process. However, Chidister and Malvern⁽²⁵⁾, Lindholm⁽²⁷⁾ and Trozera, Sherby and Dorn⁽³⁷⁾, considered the effect of a change in uniform temperature on viscoplastic behavior, with a view to confirming some results of the dislocation theory.

Appendix II

The form of eq. (3.22) for isotropic materials, according to eq.'s (3.22 a-e), is

$$\begin{aligned} \psi = & \frac{1}{2} A_1 \epsilon_{ii} \epsilon_{jj} + \frac{1}{2} A_2 \epsilon_{ij} \epsilon_{ij} + B_1^\alpha \epsilon_{ii} q_{jj}^\alpha + B_2^\alpha \epsilon_{ij} q_{ij}^\alpha \\ & + \frac{1}{2} C_1^\alpha q_{ii}^\alpha q_{jj}^\alpha + \frac{1}{2} C_2^\alpha q_{ij}^\alpha q_{ij}^\alpha + D \Theta \epsilon_{jj} + E^\alpha \Theta q_{ii}^\alpha \\ & + \frac{1}{2} F \Theta^2 \end{aligned} \quad (\text{A.2.1})$$

As a result eq.'s (3.8) and (3.9) yield:

$$\begin{aligned} \sigma_{ij} = & A_1 \delta_{ij} \epsilon_{kk} + A_2 \epsilon_{ij} + B_1^\alpha \epsilon_{ij} q_{kk}^\alpha + B_2^\alpha q_{ij}^\alpha \\ & + D \Theta \delta_{ij} \quad (\alpha \text{ summed}) \end{aligned} \quad (\text{A.2.2})$$

$$- \chi = \frac{\partial \psi}{\partial \Theta} = D \epsilon_{ii} + E^\alpha q_{ii}^\alpha + F \Theta \quad (\alpha \text{ summed}) \quad (\text{A.2.3})$$

On the other hand,

$$\begin{aligned} \frac{\partial \psi}{\partial q_{ij}^\alpha} = & B_1^\alpha \epsilon_{kk} \delta_{ij} + B_2^\alpha \epsilon_{ij} + C_1^\alpha \delta_{ij} q_{kk}^\alpha \\ & + C_2^\alpha q_{ij}^\alpha + E^\alpha \delta_{ij} \Theta \quad (\text{not summed}) \end{aligned} \quad (\text{A.2.3})$$

Hence, use of eq. (3.21) in accordance with eq. (3.23f) yields a set of first order differential equations for q_{ij}^α ; these can be expressed as a set for q_{kk}^α and another for the deviatoric part of q_{ij}^α , which we denote by p_{ij}^α . Thus in the notation of eq.'s (3.31) and (3.33)

$$B_0^\alpha \epsilon_{kk} + C_0^\alpha q_{kk}^\alpha + E^\alpha + b_0^\alpha \frac{dq_{kk}^\alpha}{dz} = 0 \quad (\text{A.2.4})$$

$$B_2^\alpha \epsilon_{ij} + C_2^\alpha p_{ij}^\alpha + b_2^\alpha \frac{dp_{ij}^\alpha}{dz} = 0 \quad (\text{A.2.5})$$

In both eq.'s (A.2.4) and (A.2.5) α is not summed. It follows from the above two equations that

$$q_{kk}^{\alpha} = - \frac{B_0^{\alpha}}{b_0^{\alpha}} \int_{z_0}^z e^{-\lambda_{\alpha}(z-z')} \epsilon_{kk}(z') dz' - \frac{E^{\alpha}}{b_0^{\alpha}} \int_{z_0}^z e^{-\lambda_{\alpha}(z-z')} \vartheta(z') dz' \quad (A.2.6)$$

$$p_{ij}^{\alpha} = - \frac{B_2^{\alpha}}{b_2^{\alpha}} \int_{z_0}^z e^{-\rho_{\alpha}(z-z')} e_{ij}(z') dz' \quad (A.2.7)$$

where λ_{α} and ρ_{α} are given by eq. (3.32).

In the light of the tensorial notation that we have adopted for the internal variables, let

$$\left| \left| \epsilon_{ij} \right| \right|_{\sup} = \Delta, \quad \left| \left| e_{ij} \right| \right|_{\sup} = \Delta_1, \quad \left| \epsilon_{kk} \right|_{\sup} = \Delta_0 \quad (A.2.8)$$

where, $\left| \left| \epsilon_{ij} \right| \right| = \left| \epsilon_{ij} \epsilon_{ij} \right|^{1/2}$, etc. Evidently,

$$\Delta^2 = \Delta_1^2 + \frac{1}{3} \Delta_0^2 \quad (A.2.9)$$

Then as a result of eq.s (A.2.7) and (A.2.8)

$$\left| \left| p_{ij}^{\alpha} \right| \right| \leq \frac{|B_2^{\alpha}|}{C_2^{\alpha}} \Delta_1 \quad (A.2.10)$$

$$\left| q_{kk}^{\alpha} \right| \leq \frac{|B_0^{\alpha}|}{C_0^{\alpha}} \Delta_0 + \frac{|E^{\alpha}|}{C_0^{\alpha}} \delta \quad (A.2.11)$$

where as before $\left| \vartheta \right|_{\sup} = \delta$

Also from eq. (A.2.5),

$$\begin{aligned} (b_2^{\alpha})^2 \left| \left| \frac{dp_{ij}}{dz} \right| \right|^2 &= (B_2^{\alpha})^2 \left| \left| e_{ij} \right| \right|^2 + (C_2^{\alpha})^2 \left| \left| p_{ij} \right| \right|^2 \\ &+ 2B_2^{\alpha} C_2^{\alpha} \left| p_{ij} e_{ij} \right| \end{aligned} \quad (A.2.12)$$

However, since

$$\left| p_{ij} e_{ij} \right| \leq \left| \left| p_{ij} \right| \right| \left| \left| e_{ij} \right| \right| \quad (A.2.13)$$

it follows from (A.2.12) that

$$b_2^a \left| \left| \frac{dp_{ij}}{dz} \right| \right| \leq 2 \Delta_1 \left| B_2^a \right| \quad (\text{A.2.14})$$

Also as a result of eq. (A.2.4)

$$b_0^a \left| \frac{dq_{kk}^a}{dz} \right| \leq 2 \left| B_0^a \right| \Delta_0 + 2 \left| E^a \right| \delta \quad (\text{A.2.15})$$

At this point we order our internal variables as shown,

$$p_{ij}^1, p_{ij}^2, \dots, p_{ij}^m; q_{kk}^1, q_{kk}^2, \dots, q_{kk}^m.$$

Let q_a be a typical internal variable. Then, whether it belongs to the p-group or the q-group above, as a result of eq.'s (A.2.10), (A.2.11), (A.2.14) and (A.2.14), given two positive members δ_1 and δ_2 , however small, we can choose Δ_0 and Δ_1 (and therefore Δ) and δ such that

$$\left| q_a \right| \leq \delta_1 \text{ and } \left| \frac{dq_a}{dz} \right| \leq \delta_2.$$

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